2[12.9] Given a curve ** in a 1-manifold **M**, a 1-form****** on **M**, coordinate patches (*x*) and (*X*) in **M**, and transition function *x* = *x*(*X*) and its inverse *X* = *X*(*x*). Let ** = *f*(*x*) *dx* in (*x*)‑coordinates. Show

 and find *A*, *B*, *F*(*X*), and *dX*.

First, a “curve” ** in **M** is simply a closed interval, say [*a*,*b*]. This shows that 

Define *A* = *X*(*a*) and *B* = *X*(*b*). So *X* = *A* when *x* = *a* and *X* = *B* when *x* = *b.*

Also, 

Thus, 

Finally,



Postscript 1. Dimbulb in his approach to this problem was concerned about the definition of *A* and *B* in the case that the relationship between *x* and *X* were not 1-1. But, that cannot be the case because whenever one has inverse functions, they are 1-1 (and onto). So there should be no concern about simply defining *A* = *X*(*a*)and *B* = *X*(*b*).

Postscript 2. This problem is insightful in many respects but in other ways its simplicity hides key insights. Let me propose and solve a simple change of variables example for a 1-form in a 2-manifold (choose **M** = ℝ2**)** to gain a larger perspective on this process. This problem has 3 givens and a bunch of unknowns.

Given:

(1) A curve ** (actually, a straight line segment) in ℝ2 defined by

*T* : [0, 1]→ ℝ2: *T*(*t*) = (*x*(*t*), *y*(*t*)) = (6*t* + 2, 3*t* + 1)

Thus,

** = *T*( [0, 1] ),

*x*(*t*) = 6*t* + 2 = 2 (3*t* + 1), and

*y*(*t*) = 3*t* + 1.

(2) A transition function: 

(3) A 1-form 

So *f*(*x*,*y*) = *x*2 and *g*(*x*, *y*) = *xy*.

Find the following:



Solution:



(b)

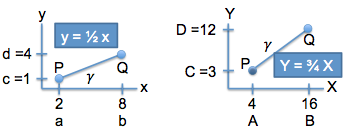
Let P and Q represent the endpoints of  in coordinate-free notation

P = (*a*, *c*) = ( *x*(0), *y*(0) ) = (2,1) and Q = (*b*, *d*) = ( *x*(1), *y*(1) ) = (8,4) in (x,y)‑coordinates.

P = (*A*, *C*) = (2*a*, *a* + *c* ) = (4, 3) and

Q = (*B*, *D*) = (2*b*, *b* + *d* ) = (16, 12) in (X,Y)‑coordinates.

The (*x*,*y*)- and (*X*,*Y*)-plots are below, along with their equations.



Now,



Thus,



(c)



 Substituting yields

